AGU PUBLICATIONS

Geophysical Research Letters

RESEARCH LETTER

10.1002/2016GL070348

Mixed-point geostatistical simulation: A combination of two- and multiple-point geostatistics

Knud Skou Cordua[1], Thomas Mejer Hansen[1], Mats Lundh Gulbrandsen[1], Christophe Barnes[2], and Klaus Mosegaard[1]

[1]Climate and Geophysics Section, Niels Bohr Institute, University of Copenhagen, Copenhagen, Denmark, [2]Department of Geosciences and Environment, University of Cergy-Pontoise, Cergy, France

Key Points:
- A theory that makes it possible to combine several 1-D conditional distributions in the same point is developed
- The developed theory is used to design a simulation algorithm that combines different types of geostatistics for different directions
- The algorithm is applied for a case using two-point statistics in horizontal direction and multiple-point statistics in vertical direction

Correspondence to:
K. S. Cordua,
knudcordua@gmail.com

Citation:
Cordua, K. S., T. M. Hansen, M. L. Gulbrandsen, C. Barnes, and K. Mosegaard (2016), Mixed-point geostatistical simulation: A combination of two- and multiple-point geostatistics, *Geophys. Res. Lett.*, *43*, 9030–9037, doi:10.1002/2016GL070348.

Received 7 JUL 2016
Accepted 26 AUG 2016
Accepted article online 31 AUG 2016
Published online 14 SEP 2016

Abstract Multiple-point-based geostatistical methods are used to model complex geological structures. However, a training image containing the characteristic patterns of the Earth model has to be provided. If no training image is available, two-point (i.e., covariance-based) geostatistical methods are typically applied instead because these methods provide fewer constraints on the Earth model. This study is motivated by the case where 1-D vertical training images are available through borehole logs, whereas little or no information about horizontal dependencies exists. This problem is solved by developing theory that makes it possible to combine information from multiple- and two-point geostatistics for different directions, leading to a mixed-point geostatistical model. An example of combining information from the multiple-point-based single normal equation simulation algorithm and two-point-based sequential indicator simulation algorithm is provided. The mixed-point geostatistical model is used for conditional sequential simulation based on vertical training images from five borehole logs and a range parameter describing the horizontal dependencies.

1. Introduction

Geostatistical methods are traditionally used for spatial estimation and simulation of geological phenomenons and are used for advanced spatial interpolations, predictions, visualizations or, more recently, as prior source of information in inversion of geophysical and remote sensing data [*Hansen et al.*, 2008; *Zhang et al.*, 2009; *Boucher*, 2009; *Irving and Singha*, 2010; *Mariethoz et al.*, 2010; *Cordua et al.*, 2012; *Ruggeri et al.*, 2013; *Jha et al.*, 2013; *Cordua et al.*, 2014]. Realistic and trustworthy geostatistical models are, therefore, crucial in, e.g., quantitative geology, natural resources evaluation, reservoir modeling, and geophysical inverse problems in order to obtain geologically feasible solutions.

Geostatistical models allow a flexible way of describing spatial dependencies of the Earth model, ranging from low-information distributions (through two-point/covariance-based geostatistics) [*Journel and Isaaks*, 1984; *Soares*, 2001] to information-rich distributions (through multiple-point geostatistics) [*Guardiano and Srivastava*, 2009; *Strebelle*, 2002]. However, until now, no geostatistical model has been developed that is capable of combining independent and considerably different spatial statistics (e.g., two- and multiple-point geostatistics) for different directions into a single geostatistical model.

Here we develop such a flexible geostatistical model for cases where the Earth model is correctly described by considerably different types of spatial information in different directions. In this way, more correct geostatistical models that are able to capture the actual knowledge available about the spatial variability can be designed without, as today, having to make use of a geostatistical model that may make implicit assumptions not accounted for by observations. Such problems may occur in all fields of geophysics relying on geostatistical models.

Information-rich complex geostatistical models based on multiple-point geostatistics rely on training images of the geological structures expected (a priori) to be found in the subsurface. The probabilistic spatial dependencies describing the geological structures can then be obtained from these images. However, in many settings such observed scenes/images cannot be provided and, instead, low-information geostatistical models based on two-point geostatistics can be applied.

In this study, we consider the case where 1-D training images describing the vertical dependencies of the subsurface are obtained from borehole logs, whereas only weak information about the horizontal dependencies exists (e.g., from the fact that multiple boreholes are present or can be inferred indirectly from seismic data [*Carpentier and Roy-Chowdhury*, 2009]). Hence, in this case, a model based on multiple-point geostatistics is suitable for describing vertical spatial dependencies, whereas the horizontal dependencies are more appropriately described by a model based on two-point geostatistics.

We develop theory that can be used to formulate a probability distribution that combines two different types of information (e.g., two-point and multiple-point) for different directions. We will refer to the distribution that combines the two- and multiple-point-based distributions as a distribution based on mixed-point geostatistics.

Finally, a sampling algorithm based on sequential simulation that is able to perform conditional sampling from combined distributions, such as the distribution based on mixed-point geostatistics, is developed and tested.

2. Combining Probability Distributions for Different Directions

Consider a set of Earth model parameters $\mathbf{m} = (m_1, m_2, \dots, m_N)^T$ associated with positions in a regular grid. For each model parameter m_i a conditional probability distribution $p(m_i|\mathbf{V})$ exists that is conditioned by a set of parameter values $\mathbf{V}$ located within some local neighborhood.

Now consider the case depicted in Figure 1 where two different conditional probability distributions $p_1(m_i \mid \mathbf{V}_1)$ and $p_2(m_i \mid \mathbf{V}_2)$ over m_i conditioned by two different sets of parameters are defined. p_1 is conditioned by parameters within a neighborhood located above and below m_i, whereas p_2 is conditioned by parameters within a neighborhood located to the left and right of m_i.

Assume that dependency between the model parameters is only known in the directions of the extend of $\mathbf{V}_1$ and $\mathbf{V}_2$ and that these dependencies are defined by $p_1(m_i \mid \mathbf{V}_1)$ and $p_2(m_i \mid \mathbf{V}_2)$, respectively. As a consequence of this, no information about the dependencies between the parameters in $\mathbf{V}_1$ and $\mathbf{V}_2$ is known and, therefore, the probability distributions over these parameters are based on mutually independent information, which is expressed as

$$p(\mathbf{V}_1, \mathbf{V}_2) = p(\mathbf{V}_1)p(\mathbf{V}_2) \tag{1}$$

The model parameter m_i is dependent on both sets of parameters $\mathbf{V}_1$ and $\mathbf{V}_2$; hence,

$$p(m_i \mid \mathbf{V}_1, \mathbf{V}_2) \neq p(m_i) \tag{2}$$

The joint probability distribution over m_i, $\mathbf{V}_1$, and $\mathbf{V}_2$ can be expressed using the following trivial relation:

$$p(m_i, \mathbf{V}_1, \mathbf{V}_2) = p(m_i)\frac{p(m_i, \mathbf{V}_1)}{p(m_i)}\frac{p(m_i, \mathbf{V}_1, \mathbf{V}_2)}{p(m_i, \mathbf{V}_1)} \tag{3}$$

Since the probability distribution over V_2 is independent of V_1 (as expressed by equation (1)), the joint distribution in equation (3) can be formulated as

$$p(m_i, \mathbf{V}_1, \mathbf{V}_2) = p(m_i)p(\mathbf{V}_1 \mid m_i)p(\mathbf{V}_2 \mid m_i) \tag{4}$$

$$= \frac{p(m_i)p(\mathbf{V}_1, m_i)p(\mathbf{V}_2, m_i)}{p(m_i)^2} \tag{5}$$

Based on equations (5) and (1), the conditional distribution over m_i given both $\mathbf{V}_1$ and $\mathbf{V}_2$ can be obtained as

$$p(m_i \mid \mathbf{V}_1, \mathbf{V}_2) = \frac{p(m_i)p(\mathbf{V}_1, m_i)p(\mathbf{V}_2, m_i)}{p(\mathbf{V}_1)p(\mathbf{V}_2)p(m_i)^2} \tag{6}$$

$$= \frac{p(m_i \mid \mathbf{V}_1)p(m_i \mid \mathbf{V}_2)}{p(m_i)} \tag{7}$$

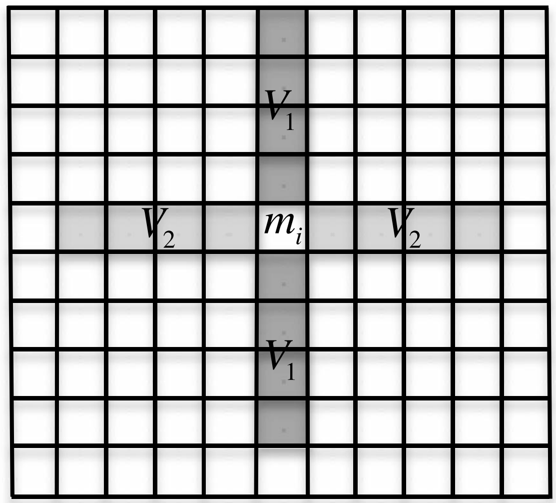

Figure 1. A regular grid of cells where each cell is associated with a parameter m_i, $i = 1, \ldots, N$. $\mathbf{V}_1$ and $\mathbf{V}_2$ are subsets of the parameters that are independent. However, both $\mathbf{V}_1$ and $\mathbf{V}_2$ are dependent on m_i.

For a general case with L nonoverlapping neighborhoods equation (7) is given as

$$p(m_i \mid \mathbf{V}_1, \mathbf{V}_2, \ldots, \mathbf{V}_L) = \frac{\prod_{k=1}^{L} p(m_i \mid \mathbf{V}_k)}{p(m_i)^{L-1}} \quad (8)$$

This equation will not be applied here but is applicable in cases of, e.g., 3-D/4-D models ($L = 3/L = 4$, with time as the fourth dimension) or for a larger set of nonoverlapping neighborhoods/conditioning events $\mathbf{V}_1, \mathbf{V}_2, \ldots, \mathbf{V}_L$.

A joint distribution over a set of parameters $p(m_1, m_2, \ldots, m_N)$ can be constructed as a product of 1-D conditional distributions over the individual parameters [*Toftaker and Tjelmeland*, 2013]

$$p(m_1, m_2, \ldots, m_N) = \prod_{i=1}^{N} p(m_i \mid \mathbf{m}_{N_i}), \quad (9)$$

where $\mathbf{m}_{N_i}$ are parameters within some neighborhood around m_i that m_i is dependent on. This distribution is also known as a partially ordered Markov model [*Toftaker and Tjelmeland*, 2013].

Now consider a sequential simulation algorithm using a local neighborhood. The joint distribution that is sampled by such an algorithm is expressed by equation (9), where $\mathbf{m}_{N_i}$ contains previously simulated parameters inside the neighborhood around m_i. Hence, the joint distribution over all parameters $\mathbf{m}$ sampled by a sequential simulation algorithm that uses equation (8) as local conditional distributions is expressed as

$$p(\mathbf{m}) = \prod_{i=1}^{N} \frac{\prod_{k=1}^{L} p\left(m_i \mid \mathbf{V}_k(m_i)\right)}{p(m_i)^{L-1}}, \quad (10)$$

where $\mathbf{V}_k(m_i)$ is the kth set of parameters that m_i is dependent on. The number of parameters contained in the individual sets $\mathbf{V}_k(m_i)$ depends on the order by which the product in equation (10) is evaluated (i.e., the random path used).

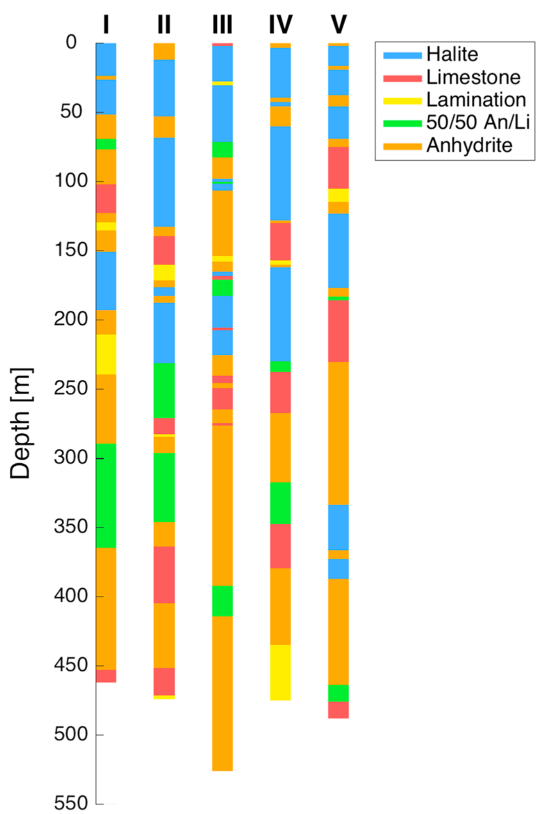

Figure 2. Five training images obtained from borehole logs, which contain information about the vertical spatial dependencies.

3. Combining Two- and Multiple-Point Geostatistics

Geostatistical simulation algorithms that are based on sequential simulation calculate and use local conditional probability distributions as part of the simulation process. Hence, two- and multiple-point (TP and MP)-based simulation algorithms are capable of providing local conditional distributions ($p_{\mathrm{TP}}(m_i \mid \mathbf{V}_1)$ and $p_{\mathrm{MP}}(m_i \mid \mathbf{V}_2)$) based on two- or multiple-point statistics, respectively.

In the example presented here, conditional distributions based on two-point geostatistics are provided by the sequential indicator simulation (SISIM) algorithm [*Journel and Alabert*, 1989; *Gómez-Hernández and Srivastava*, 1990] and conditional distributions based on multiple-point geostatistics are provided by the single normal equation simulation (SNESIM) algorithm [*Strebelle*, 2002]. Using equation (10), it is possible to formulate a joint probability distribution over the set of all parameters **m** based on a series of 1-D conditional distributions

$$p(\mathbf{m}) = \prod_{i=1}^{N} \frac{p_{\mathrm{TP}}(m_i \mid \mathbf{V}_1) p_{\mathrm{MP}}(m_i \mid \mathbf{V}_2)}{p(m_i)}. \quad (11)$$

The conditional distributions provided by SNESIM, $p_{\mathrm{MP}}(m_i \mid \mathbf{V}_1)$, is based on multiple-point statistics obtained from one-dimensional training images (e.g., see Figure 2) and describes the vertical spatial dependencies. The conditioning data event used for this distribution is located within a local neighborhood only in the same column as m_i (see Figure 1). $p_{\mathrm{TP}}(m_i \mid \mathbf{V}_2)$ is provided by SISIM and describes the horizontal dependencies via a covariance function. The conditioning event $\mathbf{V}_2$ contains parameters within a local neighborhood located only in the same row as m_i (see Figure 1). By assuming stationarity, the 1-D marginal distribution $p(m_i)$ is the same everywhere and is obtained from the training images.

4. Sequential Simulation of the Mixed-Point Geostatistical Model

The sequential simulation algorithm that samples the mixed-point geostatistical model in equation (11) takes the following steps:

1. All model parameters (i.e., pixels) are initially unrealized.
2. Visit, by random, a parameter m_i (see details below regarding the random choice).
3. Use the SISIM and SNESIM algorithms to calculate $p_{\mathrm{MP}}(m_i \mid \mathbf{V}_1)$ and $p_{\mathrm{TP}}(m_i \mid \mathbf{V}_2)$, respectively. $\mathbf{V}_1$ and $\mathbf{V}_2$ contain previously simulated parameters inside the neighborhoods located above/below and left/right relative to m_i, respectively. For some positions (including the first position) no previously simulated parameters are located within the neighborhoods. For such positions we may have that $\mathbf{V}_1 = \emptyset$ and/or $\mathbf{V}_2 = \emptyset$, where $\emptyset$ is the empty set.
4. Draw a value from
$$p(m_i \mid \mathbf{V}_1, \mathbf{V}_2) = \frac{p_{\mathrm{MP}}(m_i \mid \mathbf{V}_1) p_{\mathrm{TP}}(m_i \mid \mathbf{V}_2)}{p(m_i)} \quad (12)$$
using inverse transform sampling.
5. Repeat step 2–4 until all parameters have been visited.

In this way, a realization from equation (11) is obtained.

From information theory we have the following inequality of the entropy H of a probability distribution [*Cover and Thomas*, 2005]

$$H(m_1 \mid m_2) \leq H(m_1), \quad (13)$$

with equality if $p(m_1)$ and $p(m_2)$ are independent. $H(m_1)$ is the Shannon entropy of the distribution $p(m_1)$ [*Shannon*, 1948]. The higher entropy, the less information does the distribution contain.

Equation (13) describes that as the number of conditioning parameters increases in the local conditional distributions, the information content of the distribution increases. In order to sample from a joint distribution (equation (11)) that contains as much information as possible, the randomness of the sequences, in which the parameters are sequentially simulated, has to be designed in such a way that the number of conditioning parameters for the individual 1-D distributions will be high. Since the conditional distributions in equation (12) are only conditioned by parameters in the same column and row as m_i, the size of the conditioning events will be larger if the next parameter to be simulated is located in the same row and/or column as previously simulated parameters.

A sequential simulation path that honors this criteria is explained by Figure 3. The path is designed such that the next parameter to be simulated has as many nearby parameters as possible as conditioning parameter values. This random path has the following steps and rules:

1. The first parameter is chosen completely random among all parameters (see Figure 3 I).

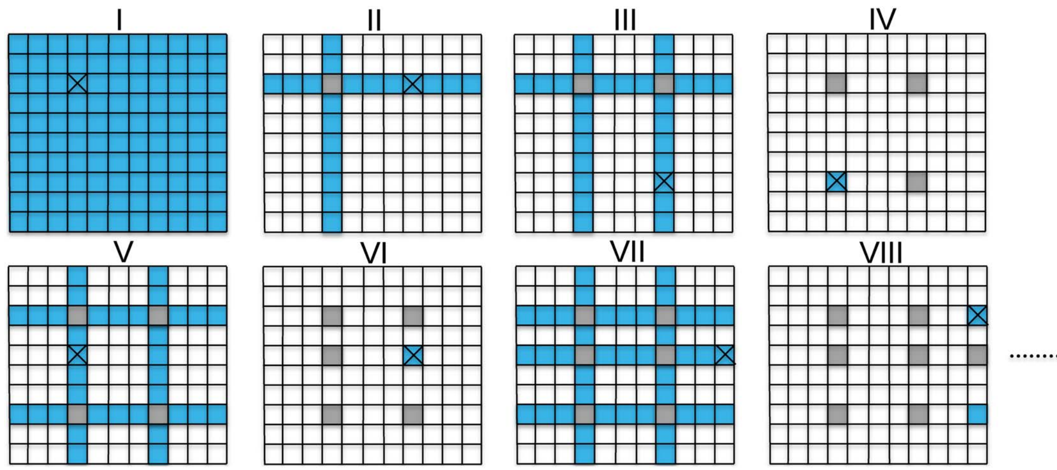

Figure 3. Eight steps of the sequential simulation algorithm using a random path conditioned on previously simulated positions. The parameter to be simulated in the individual steps, marked by a cross, is chosen randomly among the positions marked by blue. The blue positions depend on the previously simulated position in such a way that the next parameter to be simulated is conditioned by as many as possible of the previously simulated parameters in its neighborhood.

2. Hereafter, choose randomly a position that is located in the same row and/or column as a previously simulated parameter (Figure 3 II - VIII).

Step 2 may in some instances lead to only a single or a few possible positions when a row or a column of a previously simulated parameter is intersecting (e.g., Figure 3 IV, VI and VIII).

When the algorithm described in this section samples from the mixed-point geostatistical model (equation (11)), it will be referred to as a mixed-point geostatistical sequential simulation (MIXSIM) algorithm.

5. Numerical Examples

Figure 4a shows 40 independent vertical realizations from the SNESIM algorithm using single-grid sequential simulation and a template of 17 × 1 pixels. The conditional probability distribution $p_{MP}(m_i \mid \mathbf{V}_1)$ used in the

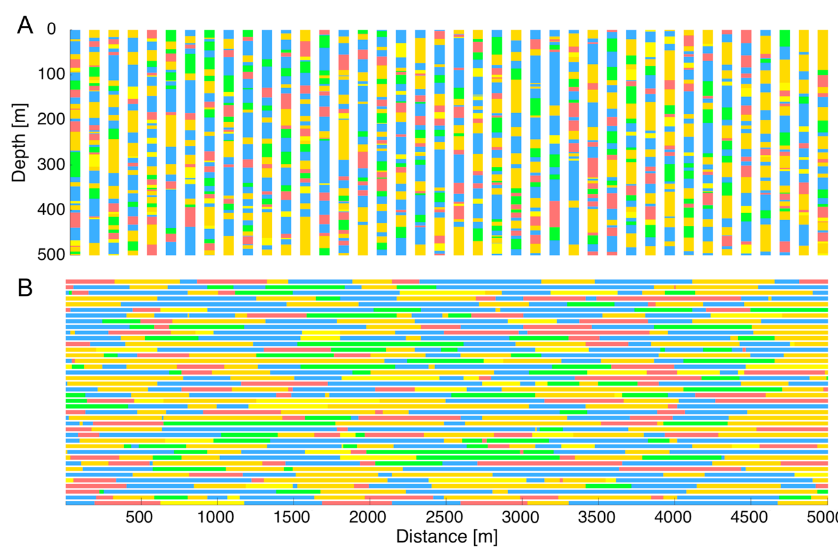

Figure 4. (a) The 40 independent 1-D vertical (500 × 1 pixels) realizations obtained through sequential simulation using $p_{MP}(m_i \mid \mathbf{V}_1)$ from SNESIM. The pattern statistics used to define p_{MP} is obtained from the training images in Figure 2. (b) The 40 independent horizontal (1 × 500 pixels) realizations obtained through sequential simulation of $p_{TP}(m_i \mid \mathbf{V}_2)$ in SISIM using a range of 50 pixels and a 1-D marginal distribution obtained from the training images.

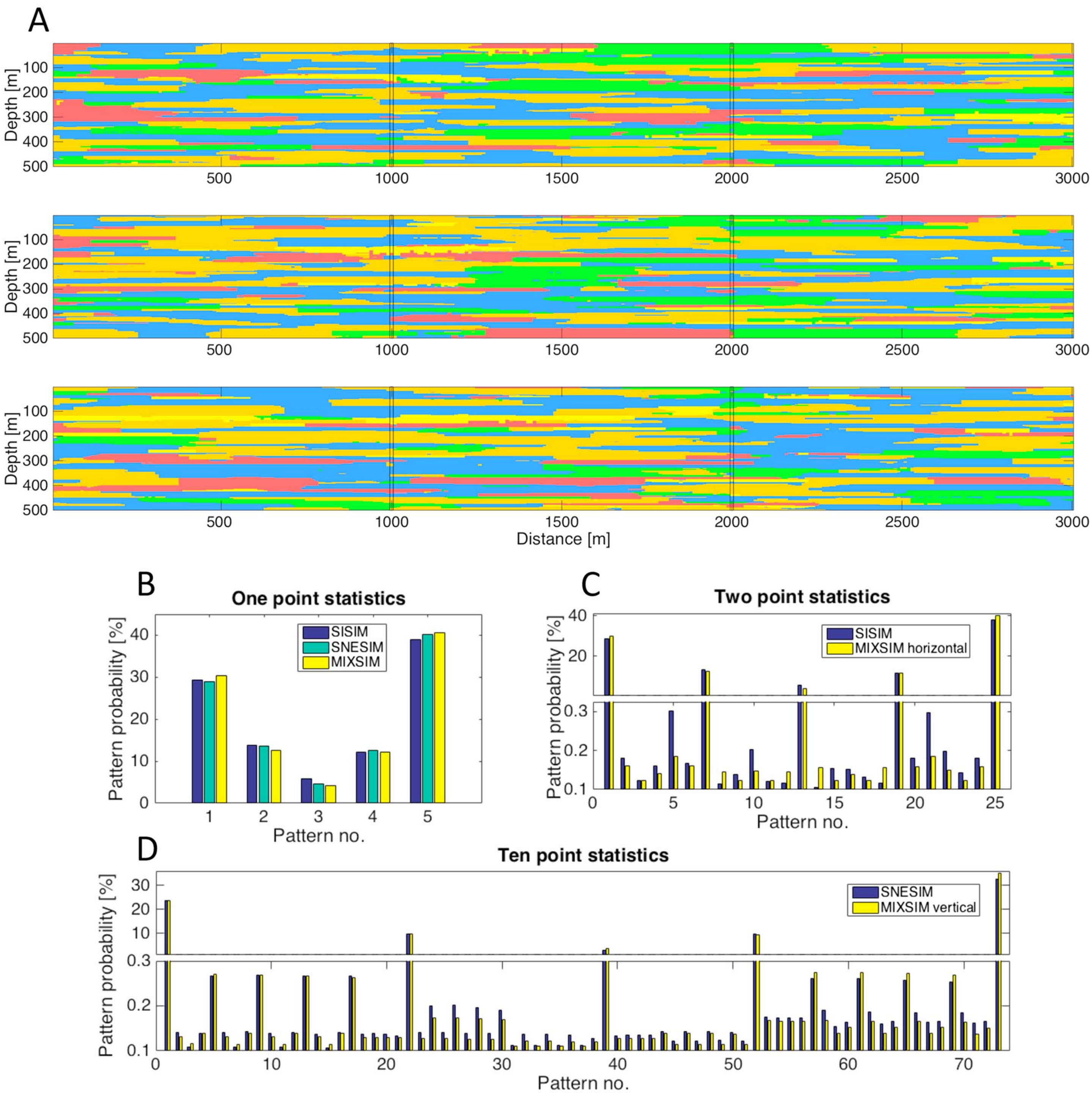

Figure 5. (a) Three realizations of 500 by 300 pixels obtained through sequential simulation of the mixed-point geostatistical model (i.e., using MIXSIM). Two boreholes (i.e., realizations from Figure 4a), located at distances $x = 1000$m and $x = 2000$m and marked by black rectangles, are used as conditioning data. (b) One-point statistics based on realizations from SISIM, SNESIM, and MIXSIM. (c) Two-point statistics based on realizations from SISIM and MIXSIM using a 1×2 pixels template. (d) Ten-point statistics based on realizations from SNESIM and MIXSIM using a 10×1 pixels template. Note the piecewise linear y axis in Figures 5c and 5d.

individual steps of the sequential simulation procedure is based on the pattern statistics obtained from the training images in Figure 2.

Figure 4b shows 40 independent horizontal realizations from the SISIM algorithm using single-grid sequential simulation. The conditional probability distribution $p_{TP}(m_i \mid \mathbf{V}_2)$ used for the sequential simulation in SISIM is based on a spherical covariance function with a range of 50 pixels. The one-dimensional marginal distribution needed by SISIM is obtained from the training images in Figure 2.

Three independent realizations obtained from the MIXSIM algorithm that combines $p_{MP}(m_i \mid \mathbf{V}_1)$ and $p_{TP}(m_i \mid \mathbf{V}_2)$ are seen in Figure 5a. The values within the black frames (in $x = 1000$ m and $x = 2000$ m) indicate positions of hard conditioning data representing two boreholes. The mixed-point geostatistical model (equation (11)) is sampled using the single-grid-based sequential simulation strategy described in section 4. A first visual comparison between the vertical and horizontal patterns in Figure 5a and the patterns in Figures 4a and 4b, respectively, indicates that the mixed-point geostatistical model has been successful in reproducing both the multiple- and two-point statistics.

Different pattern statistics have been obtained and compared in order to check, quantitatively, the reliability of the MIXSIM algorithm. Figure 5b shows a comparison between one-point statistics from realizations of the SISIM, SNESIM, and MIXSIM algorithms. Figure 5c shows a comparison of the two-point statistics produced by SISIM and MIXSIM, and Figure 5d shows a comparison of the 10-point statistics obtained from SNESIM and MIXSIM. All of the comparisons show good agreement between the tested pattern frequencies, and no significant biases are observed. However, the pattern frequencies are not matching exactly, which may be a result of MIXSIM having to satisfy two different types of statistics simultaneously.

6. Discussion

All geostatistical algorithms that are based on sequential simulation will in each step of the simulation draw a value from some 1-D marginal or conditional distribution. If two different conditional distributions $p_1(m_i \mid \mathbf{V}_1)$ and $p_2(m_i \mid \mathbf{V}_2)$ describing two different types of spatial statistics can be defined (possibly originating from two different types of geostatistical simulation algoritms), then these conditionals can be combined into a mixed-point geostatistical model using equation (7) and sampled through sequential simulation using the random path described in section 4. The only demand is that these distributions need to share the same 1-D marginal distribution $p(m_i)$ for all parameters/positions i in the grid to be simulated.

The example demonstrated in this paper is assuming noninclining strata with orthogonal extends of $\mathbf{V}_1$ and $\mathbf{V}_2$ at all locations. However, one can imagine that the algorithm can be adapted to inclined strata by locally tilting the direction of $\mathbf{V}_2$ (i.e., the horizontal part) such that it follows the local direction of the inclining structure, while $\mathbf{V}_1$ (i.e., the vertical part) follows the direction of the well.

The expressions in equations (7) and (8) are based on the implicit assumption that $p(\mathbf{V}_1)$ and $p(\mathbf{V}_2)$ are based on mutually independent information, as expressed by equation (1). According to equation (13), this is the maximum entropy (i.e., minimum information) assumption. Hence, by only knowing the dependencies in orthogonal directions implicitly involves a maximum entropy assumption in directions that deviates from the directions of known dependencies (i.e., the most information-neutral assumption is used in directions of unknown dependencies).

Examples of combining conditional distributions while assuming dependence between the conditioning events are seen in *Caers* [2006] and *Comunian et al.* [2012]. In their work, this was achieved using probability aggregation methods such as the Tau model and linear pooling methods [*Krishnan*, 2008; *Allard et al.*, 2012]. Hence, if information about the dependencies between the conditioning events $\mathbf{V}_1, \mathbf{V}_2, \ldots, \mathbf{V}_L$ are known, probability aggregation methods may provide a way of including this information [*Comunian et al.*, 2012].

7. Conclusion

The necessary theory needed in order to construct a probability distribution that combines different conditional probability distributions for different directions has been developed. This theory has been utilized to design a mixed-point geostatistical model that combines conditional distributions from two- and multiple-point-based geostatistical algorithms, and it is used to describe independent and considerably different spatial dependencies in the horizontal and vertical directions, respectively. A sequential simulation algorithm that can be used to sample the mixed-point geostatistical model has been designed. The developed mixed-point geostatistical model and the associated sampling algorithm (MIXSIM) was successfully used for conditional simulation of a 2-D field of parameters. In this case, training image-based information about the vertical dependencies was provided by borehole logs, using the SNESIM algorithm, and the horizontal dependencies was described by the SISIM algorithm based on a covariance model.

Acknowledgments
We wish to thank the Danish Innovation Foundation for providing financial support for this research. The borehole data used for this research project are provided by the Geological Survey of Denmark and Greenland (GEUS). The MIXSIM algorithm written in Matlab can be download from https://sites.google.com/site/kscordua/downloads/mixsim

References

Allard, D., A. Comunian, and P. Renard (2012), Probability aggregation methods in geoscience, *Math. Geosci.*, *44*(5), 545–581.

Boucher, A. (2009), Sub-pixel mapping of coarse satellite remote sensing images with stochastic simulations from training images, *Math. Geosci.*, *41*, 265–290.

Caers, J. (2006), A general algorithm for building 3D spatial laws from lower dimensional structural information, paper presented at 19th Stanford Center for Reservoir Forecasting Meeting, Stanford University, 8–9 May.

Carpentier, S. F. A., and K. Roy-Chowdhury (2009), Conservation of lateral stochastic structure of a medium in its simulated seismic response, *J. Geophys. Res.*, *114*, B10314, doi:10.1029/2008JB006123.

Comunian, A., P. Renard, and J. Straubhaar (2012), 3D multiple-point statistics simulation using 2D training images, *Comput. Geosci.*, *40*, 49–65.

Cordua, K. S., T. M. Hansen, and K. Mosegaard (2012), Monte Carlo full-waveform inversion of crosshole GPR data using multiple-point geostatistical a priori information, *Geophysics*, *77*(2), H19–H31, doi:10.1190/geo2011-0170.1.

Cordua, K. S., T. M. Hansen, and K. Mosegaard (2014), Improving the pattern reproducibility of multiple-point-based prior models using frequency matching, *Math. Geosci.*, *47*(3), 317–343, doi:10.1007/s11004-014-9531-4.

Cover, T. M., and J. A. Thomas (2005), *Elements of Information Theory*, Wiley-Blackwell, New York, doi:10.1002/047174882x.

Guardiano, F., and R. Srivastava (2009), Multivariate geostatistics: Beyond bivariate moments, *Geostat. Troia*, *92*(1), 133–144.

Gómez-Hernández, J. J., and R. M. Srivastava (1990), ISIM3D: An ANSI-C three-dimensional multiple indicator conditional simulation program, *Comput. Geosci.*, *16*(4), 395–440, doi:10.1016/0098-3004(90)90010-q.

Hansen, T. M., K. Mosegaard, and K. S. Cordua (2008), Using geostatistics to describe complex a priori information for inverse problems, paper presented at 8th International Geostatistics Congress, pp. 329–338.

Irving, J., and K. Singha (2010), Stochastic inversion of tracer test and electrical geophysical data to estimate hydraulic conductivities, *Water Resour. Res.*, *46*, W11514, doi:10.1029/2009WR008340.

Jha, S. K., G. Mariethoz, J. P. Evans, and M. F. McCabe (2013), Demonstration of a geostatistical approach to physically consistent downscaling of climate modeling simulations, *Water Resour. Res.*, *49*, 245–259, doi:10.1029/2012WR012602.

Journel, A., and E. Isaaks (1984), Conditional indicator simulation: Application to a Saskatchewan uranium deposit, *J. Int. Assoc. Math. Geol.*, *16*(7), 685–711, doi:10.1029/2009WR008340.

Journel, A. G., and F. G. Alabert (1989), Focusing on spatial connectivity of extreme-valued attributes: Stochastic indicator models of reservoir heterogeneities—ABSTRACT, *Bulletin*, *73*, doi:10.1306/703c9f73-1707-11d7-8645000102c1865d.

Krishnan, S. (2008), The Tau model for data redundancy and information combination in Earth sciences: Theory and application, *Math. Geosci.*, *40*(6), 705–727.

Mariethoz, G., P. Renard, and J. Caers (2010), Bayesian inverse problem and optimization with iterative spatial resampling, *Water Resour. Res.*, *46*, W11530, doi:10.1029/2010WR009274.

Ruggeri, P., J. Irving, E. Gloaguen, and K. Holliger (2013), Regional-scale integration of multiresolution hydrological and geophysical data using a two-step Bayesian sequential simulation approach, *Geophys. J. Int.*, *194*, 289–303.

Shannon, C. E. (1948), A mathematical theory of communication, *Bell Syst. Tech. J.*, *27*(3), 379–423, doi:10.1002/j.1538-7305.1948.tb01338.x.

Soares, A. (2001), Direct sequential simulation and cosimulation, *Math. Geol.*, *33*(8), 911–926.

Strebelle, S. (2002), Conditional simulation of complex geological structures using multiple-point statistics, *Math. Geol.*, *34*(1), 1–21.

Toftaker, H., and H. Tjelmeland (2013), Construction of binary multi-grid Markov random field prior models from training images, *Math. Geosci.*, *45*(4), 383–409.

Zhang, X., M. J. Pyrcz, and C. V. Deutch (2009), Stochastic surface modeling of deepwater depositional systems for improved reservoir models, *J. Pet. Sci. Eng.*, *68*, 118–134.